

**NUMERICAL SOLUTION OF FIFTH ORDER BVP BY GALERKIN
METHOD WITH CUBIC B-SPLINES**

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Abstract: We developed a numerical method to solve a general fifth order two point bvp using Galerkin method with cubic B-splines. The basis functions are modified into a new set of basis functions which vanish on the all most all boundary conditions. To test the efficiency of the proposed method, we applied this scheme on several fifth order linear and nonlinear bvp's. The solution of a non-linear boundary value problem has been obtained by quasilinearization technique. We found numerical results were closed to the exact solutions available in the literature.

Keywords and Phrases: Galerkin method, Cubic B-spline, Fifth order bvp, Error.

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1. Introduction and Preliminaries

We consider a general fifth order linear bvp

$$a_0(x)y^{(5)}(x)+a_1(x)y^{(4)}(x)+a_2(x)y'''(x)+a_3(x)y''(x)+a_4(x)y'(x)+a_5(x)y(x)=b(x), \\ c < x < d \quad (1)$$

subject to boundary conditions

$$y(c)=A_0, \quad y(d)=C_0, \quad y'(c)=A_1, \quad y'(d)=C_1, \quad y''(c)=A_2 \quad (2)$$

where A'_i, B'_i 's are finite real constants and $a_i(x)$'s, $b(x)$, $i = 0, 1, \dots, 5$ are all continuous functions over $[c, d]$.

Generally, these kinds of fifth order bvp's arise frequently in the mathematical modeling of viscoelastic fluids and other branches of mathematics, physical and engineering sciences [1, 2]. The detail of existence and uniqueness of the solution for these problems can be found in the monograph namely Agarwal [3]. Solving such boundary value problems analytically is not always possible and it is only possible in very few cases only. The numerical study of these equations has been challenging by their nature and is in growing interest to researchers. So, many numerical methods have been developed over the years to approximate the solution for these types of boundary value problems. Some of the already established methods are Finite Difference Method(FDM), Homotopy Analysis Method(HAM), Optimal HAM, Collocation method, VPM, VIM, HPM, Decomposition method, Sinc Galerkin method, Adomain decomposition method etc.

We mainly focus on the solving fifth order bvp's by using spline functions techniques. Spectral Galerkin and Spectral Collocation methods were developed by Davies A. R et al. [1, 2] to solve fifth order bvp, A special type of fifth order bvp's solved by Caglar et al. and Kasi Viswanadham et al. [4, 5] used spline technique Collocation methods with B-splines sixth degree polynomials respectively. The quintic B-spline Galerkin method used to solve special case of fifth order bvp's by Kasi Viswanadham and Murali Krishna [6], Siddiqi et al. and Kasi Viswanadham, Showri Raju [7, 8, 9] developed spline technique with quartic polynomial and cubic, quartic B-spline Collocation methods respectively to solve fifth order boundary value problems. Feng-Gong Lang and Xiao-Ping Xu [10, 11] developed the Collocation method based on FDM to solve a special case of fifth order bvp's by using cubic and quartic B-splines. This motivated us to solve a general fifth order bvp by Galerkin method with cubic B-splines.

2. Justification for using Galerkin Method

In the Galerkin method, the residual of approximation is made orthogonal to the basis functions. When we use the Galerkin method, a weak form of approximation solution for a given differential equation exists and is unique under appropriate conditions [12, 13] irrespective of properties of a given differential operator. Further, a weak solution also tends to a classical solution of a given differential equation, provided sufficient attention is given to boundary conditions [14]. That means the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are prescribed. Hence in this paper we employed the use of Galerkin method with cubic B-splines as basis functions to approximate the solution of fifth order bvp's.

3. Description of the Method

Definition of cubic B-splines. The cubic B-splines $B_i(x)$'s are defined by

$$B_i(x) = \begin{cases} \sum_{r=i-2}^{i+2} \frac{(x_r - x)_+^3}{\pi'(x_r)}, & \text{for } x \in [x_{i-2}, x_{i+2}] \\ 0, & \text{otherwise} \end{cases}$$

where $\{B_{-1}(x), B_0(x), B_1(x), \dots, B_{n-1}(x), B_n(x), B_{n+1}(x)\}$ forms a basis for the space $S_3(\pi)$ of a cubic polynomial splines. More details available in the monographs [15, 16]. Schoenberg [16] has proved that cubic B-splines are the unique nonzero splines of smallest compact support $[x_{-3}, x_{n+3}]$. The approximate solution $y(x)$ of (1) is defined as

$$y(x) = \sum_{j=-1}^{n+1} \alpha_j B_j(x) \quad (3)$$

where α_j 's are the nodal parameters. Since we are approximating the fifth order bvp by cubic B-spline polynomial, we redefine the basis functions into a new set of basis functions which vanish on Dirichlet and Neumann boundary conditions. The procedure for redefining the basis functions is as follows.

Using the definition of cubic B-splines and the boundary conditions of (2), we get the approximate the boundary conditions as

$$A_0 = y(c) = y(x_0) = \alpha_{-1}B_{-1}(x_0) + \alpha_0B_0(x_0) + \alpha_1B_1(x_0) \quad (4)$$

$$C_0 = y(d) = y(x_n) = \alpha_{n-1}B_{n-1}(x_n) + \alpha_nB_n(x_n) + \alpha_{n+1}B_{n+1}(x_n) \quad (5)$$

$$A_1 = y'(c) = y'(x_0) = \alpha_{-1}B'_{-1}(x_0) + \alpha_0B'_0(x_0) + \alpha_1B'_1(x_0) \quad (6)$$

$$C_1 = y'(d) = y'(x_n) = \alpha_{n-1}B'_{n-1}(x_n) + \alpha_nB'_n(x_n) + \alpha_{n+1}B'_{n+1}(x_n) \quad (7)$$

Eliminating $\alpha_0, \alpha_1, \alpha_n$ and α_{n+1} from the (4) - (7), we get approximation $y(x)$ as

$$y(x) = w(x) + \sum_{j=1}^{n-1} \alpha_j \tilde{B}_j(x) \quad (8)$$

where

$$w(x) = w_1(x) + \frac{A_1 - w'_1(x_0)}{P'_0(x_0)} P_0(x) + \frac{C_1 - w'_1(x_n)}{P'_n(x_n)} P_n(x)$$

$$w_1(x) = \frac{A_0}{B_{-1}(x_0)}B_{-1}(x) + \frac{C_0}{B_{n+1}(x_n)}B_{n+1}(x)$$

$$\tilde{B}_j(x) = \begin{cases} P_j(x) - \frac{P'_j(x_0)}{P'_0(x_0)}P_0(x), & j = 1 \\ P_j(x), & j = 2, 3, \dots, n-2 \\ P_j(x) - \frac{P'_j(x_n)}{P'_n(x_n)}P_n(x), & j = n-1. \end{cases}$$

$$P_j(x) = \begin{cases} B_j(x) - \frac{B_j(x_0)}{B_{-1}(x_0)}B_{-1}(x), & j = 0, 1 \\ B_j(x), & j = 2, 3, \dots, n-2 \\ B_j(x) - \frac{B_j(x_n)}{B_{n+1}(x_n)}B_{n+1}(x), & j = n-1, n. \end{cases}$$

Applying the Galerkin method to (1) with the modified basis functions $\{\tilde{B}_j(x), j = 1, 2, \dots, n-1\}$, we get

$$\int_{x_0}^{x_n} [a_0(x)y^{(5)}(x) + a_1(x)y^{(4)}(x) + a_2(x)y'''(x) + a_3(x)y''(x) + a_4(x)y'(x) + a_5(x)y(x)] \tilde{B}_i(x) dx = \int_{x_0}^{x_n} b(x)\tilde{B}_i(x)dx \quad \text{for } i = 1, 2, \dots, n-1. \quad (9)$$

Integrating by parts the first three terms on the left hand side of (9) and after applying the boundary conditions prescribed in (2), we obtained the system of linear equations and these were arranged in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \quad (10)$$

where $\mathbf{A} = [a_{ij}]$;

$$a_{ij} = \int_{x_0}^{x_n} \left\{ \left[-\frac{d^3}{dx^3} [a_0(x)\tilde{B}_i(x)] - \frac{d}{dx} [a_2(x)\tilde{B}_i(x)] + a_3(x)\tilde{B}_i(x) \right] \tilde{B}_j''(x) \right. \\ \left. - \frac{d}{dx} [a_1(x)\tilde{B}_i(x)] \tilde{B}_j'''(x) + a_4(x)\tilde{B}_i(x)\tilde{B}_j'(x) + a_5(x)\tilde{B}_i(x)\tilde{B}_j(x) \right\} dx \\ + \left[\frac{d^2}{dx^2} [a_0(x)\tilde{B}_i(x)] \tilde{B}_j''(x) \right]_{x_n} \quad \text{for } i = 1, 2, \dots, n-1, \quad j = 1, 2, \dots, n-1 \quad (11)$$

$$\mathbf{B} = [b_i];$$

$$\begin{aligned} b_i = & \int_{x_0}^{x_n} \left\{ b(x) \tilde{B}_i(x) + \left[\frac{d^3}{dx^3} [a_0(x) \tilde{B}_i(x)] + \frac{d}{dx} [a_2(x) \tilde{B}_i(x)] - a_3(x) \tilde{B}_i(x) \right] w''(x) \right. \\ & \left. + \frac{d}{dx} [a_1(x) \tilde{B}_i(x)] w'''(x) - a_4(x) \tilde{B}_i(x) w'(x) - a_5(x) \tilde{B}_i(x) w(x) \right\} dx \\ & + A_2 \frac{d^2}{dx^2} [a_0(x) \tilde{B}_i(x)] \Big|_{x_0} - \left[\frac{d^2}{dx^2} [a_0(x) \tilde{B}_i(x)] w''(x) \right]_{x_n} \quad \text{for } i = 1, 2, \dots, n-1 \end{aligned} \quad (12)$$

and $\alpha = [\alpha_1 \ \alpha_2 \ \dots \ \alpha_{n-1}]^T$ and this α has been obtained from the solving seven diagonal band system $\mathbf{A}\alpha = \mathbf{B}$ using a band matrix solution package.

4. Numerical Results

To demonstrate the applicability of the proposed numerical method we consider a linear and a nonlinear fifth order bvp's. We compare the numerical results which are obtained by the proposed method with the exact solutions and these are presented in the table form. We used uniform norm to measure the error in each example and we consider modest step size $h = 0.1$.

Example 1. Consider the linear bvp

$$y^{(5)} + xy = (1-x) \cos x - 5 \sin x + x \sin x - x^2 \sin x, \quad 0 < x < 1 \quad (13)$$

subject to $y(0) = 0$, $y(1) = 0$, $y'(0) = 1$, $y'(1) = -\sin 1$, $y''(0) = -2$.

The exact solution for the above problem is $y = (1-x) \sin x$. The obtained numerical results for this problem are given in Table 1. The error obtained by the proposed method is 4.202127×10^{-6} .

Example 2. Consider the nonlinear bvp

$$y^{(5)} + [y']^2 e^{4y} - 4y^2 e^{y''} + e^{2x} [y''']^2 = 32e^{-2x}, \quad 0 < x < 1 \quad (14)$$

subject to $y(0) = 1$, $y(1) = e^{-2}$, $y'(0) = -2$, $y'(1) = -2e^{-2}$, $y''(0) = 4$.

The exact solution for the above problem is $y = e^{-2x}$. The nonlinear boundary value problem (14) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [17] as

$$\begin{aligned} y_{(n+1)}^{(5)} + (2e^{2x} y_{(n)}''') y_{(n+1)}''' - (4[y_{(n)}]^2 e^{y_{(n)}''}) y_{(n+1)}'' + (2y_{(n)}' e^{4y_{(n)}}) y_{(n+1)}' + (4[y_{(n)}']^2 e^{4y_{(n)}}) \\ - 8y_{(n)} e^{y_{(n)}''}) y_{(n+1)} = 32e^{-2x} + e^{4y_{(n)}} [y_{(n)}']^2 - 4[y_{(n)}]^2 e^{y_{(n)}''} + e^{2x} [y_{(n)}''']^2 + 4y_{(n)} [y_{(n)}']^2 e^{4y_{(n)}} \\ - 4[y_{(n)}]^2 y_{(n)}'' e^{y_{(n)}''}, \quad n = 0, 1, 2, \dots \end{aligned} \quad (15)$$

$y_{(n+1)}(0) = 1$, $y_{(n+1)}(1) = e^{-2}$, $y'_{(n+1)}(0) = -2$, $y'_{(n+1)}(1) = -2e^{-2}$, $y''_{(n+1)}(0) = 4$. The proposed method is applied to the sequence of linear problems (15). The obtained numerical results for this problem are presented in Table 2. The error obtained by the proposed method is 7.694364×10^{-4} .

Numerical results for Example 1			Numerical results for Example 2		
0.1	8.985008E-02	3.121793E-06	0.1	8.187308E-01	3.808737E-05
0.2	1.589355E-01	3.129244E-07	0.2	6.703200E-01	2.675056E-04
0.3	2.068641E-01	3.665686E-06	0.3	5.488116E-01	4.627109E-04
0.4	2.336510E-01	1.028180E-06	0.4	4.493290E-01	4.867315E-04
0.5	2.397128E-01	4.202127E-06	0.5	3.678795E-01	1.728833E-04
0.6	2.258570E-01	1.206994E-06	0.6	3.011942E-01	3.091395E-04
0.7	1.932653E-01	4.127622E-06	0.7	2.465970E-01	7.694364E-04
0.8	1.434712E-01	7.748604E-07	0.8	2.018965E-01	8.221567E-04
0.9	7.833266E-02	3.330410E-06	0.9	1.652989E-01	4.288703E-04

5. Conclusion

In this paper we have developed the numerical scheme which involves a finite element Galerkin approach with cubic B-splines as basis functions to solve a general fifth order bvp. From these numerical results we conclude that these numerical results are close to the exact solutions available in the literature. The objective of this paper is to present a simple and efficient method to solve a general fifth order boundary value problem.

References

- [1] Davies A. R., Karageorghis and Phillips, Spectral Galerkin Methods for the primary two point boundary value problem in modeling viscoelastic flows, International Journal of Numerical Methods in Engineering, 26, 647-662, (1988).
- [2] Davies A. R., Karageorghis and Phillips, Spectral Collocation Methods for the primary two point boundary value problem in modeling viscoelastic flows, International Journal of Numerical Methods in Engineering, 26, 805-813, (1988).
- [3] Agarwal R. P., Boundary value problems for Higher Order Differential Equations, World Scientific, Singapore, (1986).
- [4] Hikmet Caglar, Nazan Caglar and Twizell, The Numerical Solution of fifth order boundary value problems with sixth degree B-spline functions, Applied Mathematics Letter, 12, 25-30, (1999).

- [5] Kasi Viswanadham K. N. S. and Murali Krishna P. and Prabhakara Rao C., Numerical Solution of Fifth Order Boundary Value Problems by Collocation Method with Sixth Order B-Splines, *International Journal of Applied Science and Engineering*, 8, 119-125, (2010).
- [6] Kasi Viswanadham K. N. S. and Murali Krishna P., Quintic B-splines Galerkin for fifth order boundary value problems, *ARNP Journal of Engineering and Applied Sciences*, 5, 74-77, (2010).
- [7] Shahid S. Siddiqi and Ghazala Akram and Arfa Elahi, Quartic spline solutions of linear fifth order boundary value problems, *Applied Mathematics and Computations*, 196, 214-220, (2008).
- [8] Kasi Viswanadham K. N. S. and Showri Raju Y., Cubic B-Spline Collocation Method for fifth boundary value problems, *International Journal for Mathematical Sciences and Engineering Applications*, 6, 431-447, (2012).
- [9] Kasi Viswanadham K. N. S. and Showri Raju Y., Quartic B-Spline Collocation Method for fifth boundary value problems, *International Journal of Computer Applications*, 43, 1-6, (2012).
- [10] Feng-Gong Lang and Xiao-Ping Xu, A new cubic B-Spline method for linear fifth order boundary value problems, *Journal of Computational and Applied Mathematics*, 36, 101-116, (2011).
- [11] Feng-Gong Lang and Xiao-Ping Xu, Quartic B-Spline collocation method for fifth order boundary value problems, *Computing*, 92, 365-378, (2011).
- [12] Bers L, John F. and Schechter M., *Partial Differential Equations*, John Wiley Inter Science, New York, (1964).
- [13] Lions J. L. and Magenes E., *Non-Homogeneous Boundary Value Problem and Applications*, Springer-Verlag, Berlin, (1972).
- [14] Mitchel A. R. and Wait R., *The Finite Element Method in Partial Differential Equations*, John Wiley and Sons, London, (1977).
- [15] Prenter P. M., *Splines and Variational Methods*, John-Wiley and Sons, New York, (1989).
- [16] Schoenberg I. J., *On Spline Functions*, MRC Report 625, University of Wisconsin, (1966).

- [17] Bellman R. E. and Kalabaa R. E., Quasilinearzation and Nonlinear Boundary Value Problems, American Elsevier, New York, (1965).